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Properties of the Extended Whittaker Function

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Abstract: In this article, we define an extended form of the Whittaker function by using extended confluent hypergeometric function of the first kind and study several of its properties. We also define the extended confluent hypergeometric function of the second kind and show that this function occurs naturally in statistical distribution theory.

Key words: Beta function; Extended beta function; Extended confluent hypergeometric function; Extended gamma function; Extended Gauss hypergeometric function; Gamma distribution; Gauss hypergeometric function

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1. INTRODUCTION

The classical beta function, denoted by $B(a,b)$, is defined (see Luke [8]) by the Euler's integral

$$\begin{aligned} B(a,b) &= \int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t, \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad \mathrm{Re}(a)>0, \quad \mathrm{Re}(b)>0. \end{aligned} \tag{1}$$

The Gauss hypergeometric function, denoted by $F(a,b;c;z)$, and the Kummer's function or the confluent hypergeometric function the first kind, denoted by $\Phi(b;c;z)$, for $\mathrm{Re}(c)>\mathrm{Re}(b)>0$, are defined as (see Luke [8]),

$$F(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a}\,\mathrm{d}t, \quad |\arg(1-z)|<\pi, \tag{2}$$

and

$$\Phi(b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\exp(zt)\,\mathrm{d}t. \tag{3}$$

Using the series expansions of $(1-zt)^{-a}$ and $\exp(zt)$ in (2) and (3), respectively, the series representations of the hypergeometric functions can be obtained as

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n B(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \quad |z|<1, \quad \mathrm{Re}(c)>\mathrm{Re}(b)>0, \tag{4}$$

$$\Phi(b;c;z) = \sum_{n=0}^{\infty} \frac{B(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \quad \mathrm{Re}(c)>\mathrm{Re}(b)>0. \tag{5}$$

From the confluent hypergeometric function of the first kind $\Phi(b;c;z)$, the Whittaker function $M_{\kappa,\mu}(z)$ is defined as (Whittaker and Watson [13]),

$$M_{\kappa,\mu}(z) = z^{\mu+1/2}\exp\left(-\frac{z}{2}\right)\Phi\left(\mu-\kappa+\frac{1}{2};2\mu+1;z\right), \tag{6}$$

where $\mathrm{Re}(\mu)>-1/2$ and $\mathrm{Re}(\mu\pm\kappa)>-1/2$. The Whittaker function is a special solution of Whittaker's equation, a modified form of the confluent hypergeometric equation introduced by Whittaker [12] to make the formulas involving the solutions more symmetric.

Chaudhry *et al.* [2] extended the classical beta function to the whole complex plane by introducing in the integrand of the integral given in (1) the exponential factor $\exp[-\sigma/t(1-t)]$, with $\sigma>0$. Thus, the extended beta function is defined as

$$B(a,b;\sigma) = \int_0^1 t^{a-1}(1-t)^{b-1}\exp\left[-\frac{\sigma}{t(1-t)}\right]\mathrm{d}t, \quad \sigma>0. \tag{7}$$

If we take $\sigma=0$ in (7), then for $\mathrm{Re}(a)>0$ and $\mathrm{Re}(b)>0$ we have $B(a,b;0)=B(a,b)$. Further, replacing t by $1-t$ in (7), one can see that $B(a,b;\sigma)=B(b,a;\sigma)$. The rational and justification for introducing this function are given in Chaudhry *et al.* [2] where several properties and a statistical application have also been studied. Miller [9] further studied this function and has given several additional results.

In 2004, Chaudhry *et al.* [3] gave definitions of the extended Gauss hypergeometric function and the extended confluent hypergeometric function of the first kind, denoted by $F_\sigma(a,b;c;z)$ and $\Phi_\sigma(b;c;z)$, respectively. These definitions were developed by considering the extended beta function (7) instead of beta function (1) that appears in the general term of the series (4) and (5). They suggested

$$F_\sigma(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n B(b+n,c-b;\sigma)}{B(b,c-b)} \frac{z^n}{n!}, \;\; \sigma \geq 0, \;\; |z|<1, \;\; \mathrm{Re}(c) > \mathrm{Re}(b) > 0, \tag{8}$$

$$\Phi_\sigma(b;c;z) = \sum_{n=0}^{\infty} \frac{B(b+n,c-b;\sigma)}{B(b,c-b)} \frac{z^n}{n!}, \;\; \sigma \geq 0, \quad \mathrm{Re}(c) > \mathrm{Re}(b) > 0. \tag{9}$$

Further, using the integral representation of the extended beta function (7) in (8) and (9), Chaudhry *et al.* [3] obtained integral representations, for $\sigma \geq 0$ and $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, of the extended Gauss hypergeometric function and the extended confluent hypergeometric function of the first kind as

$$F_\sigma(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} \exp\left[-\frac{\sigma}{t(1-t)}\right] \mathrm{d}t, \\ |\arg(1-z)| < \pi, \tag{10}$$

and

$$\Phi_\sigma(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \exp\left[zt - \frac{\sigma}{t(1-t)}\right] \mathrm{d}t, \tag{11}$$

respectively.

In this article, we define the extended form of the Whittaker function and derive several results pertaining to it. We also define the extended confluent hypergeometric function of the second kind.

In Section 2, several known properties of the extended beta, extended Gauss hypergeometric and extended confluent hypergeometric functions have been given. The extended Whittaker function and its properties are given in Section 3. Finally, in Section 4, the extended confluent hypergeometric function of the second kind is defined and an application of this function to statistical distributions is also given.

2. SOME KNOWN DEFINITIONS AND RESULTS

We shall begin by briefly reviewing some of the definitions and basic properties of special function and statistical distributions that will be useful in our later work.

An integral representation of the modified Bessel function of the second kind (Gradshteyn and Ryzhik [4, Eq. 3.471.9]) is given by

$$K_\nu(2\sqrt{ab}) = \frac{1}{2}\left(\frac{a}{b}\right)^{\nu/2} \int_0^\infty t^{\nu-1} \exp\left[-\left(at + \frac{b}{t}\right)\right] \mathrm{d}t, \tag{12}$$

where $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(b) > 0$. It can easily be noticed that the extended gamma function (Chaudhry and Zubair [1], Nagar, Roldán-Correa and Gupta [10]) is very similar to the modified Bessel function of the second kind defined above. In fact

$$\Gamma(\delta;\sigma) = 2\sigma^{\delta/2} K_\delta(2\sqrt{\sigma}), \tag{13}$$

where, for $\sigma > 0$ and an arbitrary complex number δ, the extended gamma function, denoted by $\Gamma(\delta;\sigma)$, is defined by

$$\Gamma(\delta;\sigma) = \int_0^\infty t^{\delta-1} \exp\left[-\left(t + \frac{\sigma}{t}\right)\right] \mathrm{d}t.$$

From the definition of the extended gamma function, it is clear that, if $\sigma = 0$, then for $\mathrm{Re}(\delta) > 0$, the extended gamma function reduces to an ordinary gamma function $\Gamma(\delta)$.

Next, we give several properties of the extended beta, extended Gauss hypergeometric, and extended confluent hypergeometric functions. These results have been taken from Chaudhry *et al.* [2,3].

If we consider $z = 1$ in (10) and compare the resulting expression with the representation (7), we find that the extended beta function and the extended Gauss hypergeometric function are related by the expression

$$F_\sigma(a,b;c;1) = \frac{B(b, c-b-a;\sigma)}{B(b,c-b)}, \quad \mathrm{Re}(c) > \mathrm{Re}(b) > 0. \tag{14}$$

In the integral representation of the extended confluent hypergeometric function (11) consider the substitution $1-u = t$, whose Jacobian is given by $J(t \to u) = 1$, to obtain

$$\Phi_\sigma(b;c;z) = \frac{\exp(z)}{B(b,c-b)} \int_0^1 (1-u)^{b-1} u^{c-b-1} \exp\left[-zu - \frac{\sigma}{u(1-u)}\right] \mathrm{d}u. \tag{15}$$

By evaluating the integral in (15) using (11), the Kummer's relation for the extended confluent hypergeometric function of the fist kind is derived as

$$\Phi_\sigma(b;c;z) = \exp(z)\Phi_\sigma(c-b;c;-z). \tag{16}$$

For $\sigma = 0$, the expression (16) reduces to the well known Kummer's first formula for the classical confluent hypergeometric function.

The Mellin transform of the extended confluent hypergeometric function is given by

$$\int_0^\infty \sigma^{s-1} \Phi_\sigma(b;c;z)\,\mathrm{d}\sigma = \frac{\Gamma(s) B\left(b+s, c-b+s\right)}{B\left(b, c-b\right)} \Phi(b+s; c+2s; z), \tag{17}$$

where $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s+b) > 0$ and $\mathrm{Re}(s+c-b) > 0$.

Finally, we define the gamma and extended beta type 2 distributions. These definitions can be found in Johnson, Kotz and Balakrishnan [7] and Nagar and Roldán-Correa [11].

Definition 2.1. *A random variable X is said to have a gamma distribution with parameters $\theta\,(>0), \kappa\,(>0)$, denoted by $X \sim \mathrm{Ga}(\kappa,\theta)$, if its probability density function (pdf) is given by*

$$\frac{x^{\kappa-1}\exp\left(-x/\theta\right)}{\theta^\kappa \Gamma(\kappa)}, \quad x > 0. \tag{18}$$

Definition 2.2. *A random variable V is said to have an extended beta type 2 distribution with parameters (p, q, σ), denoted by $V \sim \mathrm{EB2}(p, q; \sigma)$, if its pdf is given by*

$$\frac{v^{p-1}(1+v)^{-(p+q)}}{B(p,q;\sigma)\exp(2\sigma)}\exp\left[-\sigma\left(v+\frac{1}{v}\right)\right], \quad v>0. \tag{19}$$

where $B(p, q; \sigma)$ is the extended beta function defined by (7), $\sigma > 0$, and $-\infty < p, q < \infty$.

For $\sigma = 0$ with $a > 0$ and $b > 0$, the density (19) reduces to a beta type 2 density.

The matrix variate generalizations of the gamma and extended beta type 2 distributions are given in Gupta and Nagar [5], Iranmanesh *et al.* [6], and Nagar and Roldán-Correa [11].

3. EXTENDED WHITTAKER FUNCTION

This section gives the definition of the extended Whittaker function, which is an extended form of the Whittaker function (6). Several properties and integral representations of this function are also derived.

Definition 3.1. *The extended Whittaker function, denoted by $M_{\sigma,\kappa,\mu}(z)$, for $\sigma \geq 0$, is defined as*

$$M_{\sigma,\kappa,\mu}(z) = z^{\mu+1/2}\exp\left(-\frac{z}{2}\right)\Phi_\sigma\left(\mu-\kappa+\frac{1}{2};2\mu+1;z\right), \tag{20}$$

where $\mathrm{Re}(\mu) > -1/2$, $\mathrm{Re}(\mu \pm \kappa) > -1/2$ and Φ_σ is the extended confluent hypergeometric function of the first kind defined by (11).

If we consider $\sigma = 0$ in (20), then the extended Whittaker function reduces to the classical Whittaker function, *i.e.*, $M_{0,\kappa,\mu}(z) = M_{\kappa,\mu}(z)$.

An integral representation for the extended Whittaker function $M_{\sigma,\kappa,\mu}(z)$ can be obtained by replacing extended confluent hypergeometric function in (20) by its integral representation (11). Thus, we get

$$\begin{aligned} M_{\sigma,\kappa,\mu}(z) &= \frac{z^{\mu+1/2}\exp(-z/2)}{B(\mu-\kappa+1/2,\mu+\kappa+1/2)} \\ &\quad\times\int_0^1 t^{\mu-\kappa-1/2}(1-t)^{\mu+\kappa-1/2}\exp\left[zt-\frac{\sigma}{t(1-t)}\right]\mathrm{d}t. \end{aligned} \tag{21}$$

Using the transformation $t = (u - \alpha)/(\beta - \alpha)$ with the Jacobian $(\beta - \alpha)^{-1}$ in (21), the extended Whittaker function can also be represented as

$$\begin{aligned} M_{\sigma,\kappa,\mu}(z) &= \frac{(\beta-\alpha)^{-2\mu}z^{\mu+1/2}\exp(-z/2)}{B(\mu-\kappa+1/2,\mu+\kappa+1/2)}\int_\alpha^\beta (u-\alpha)^{\mu-\kappa-1/2}(\beta-u)^{\mu+\kappa-1/2} \\ &\quad\times\exp\left[\frac{z(u-\alpha)}{\beta-\alpha}-\frac{\sigma(\beta-\alpha)^2}{(u-\alpha)(\beta-u)}\right]\mathrm{d}u, \end{aligned} \tag{22}$$

where α and β are two scalars such that $\beta - \alpha > 0$. If we consider $\beta = 1$ and $\alpha = -1$ in (22), we have another integral representation of the extended extended

Whittaker function as

$$M_{\sigma,\kappa,\mu}(z) = \frac{2^{-2\mu} z^{\mu+1/2}}{B(\mu-\kappa+1/2,\mu+\kappa+1/2)} \times \int_{-1}^{1} (1+u)^{\mu-\kappa-1/2}(1-u)^{\mu+\kappa-1/2} \exp\left(\frac{zu}{2}-\frac{4\sigma}{1-u^2}\right) \mathrm{d}u. \quad (23)$$

In the integral representation of the extended Whittaker function given in (21), substituting $t=(1+u)^{-1}u$, with the Jacobian $J(t\to u)=(1+u)^{-2}$, alternative integral representation is obtained as

$$M_{\sigma,\kappa,\mu}(z) = \frac{\exp(-2\sigma) z^{\mu+1/2}\exp\left(-z/2\right)}{B(\mu-\kappa+1/2,\mu+\kappa+1/2)} \times \int_{0}^{\infty} \frac{u^{\mu-\kappa-1/2}}{(1+u)^{2\mu+1}} \exp\left[\frac{zu}{1+u}-\sigma\left(u+\frac{1}{u}\right)\right] \mathrm{d}u. \quad (24)$$

Clearly, when we take $\sigma=0$ in (21), (22), (23) and (24), we obtain integral representations of classical Whittaker function, namely,

$$M_{\kappa,\mu}(z) = \frac{z^{\mu+1/2}\exp(-z/2)}{B(\mu-\kappa+1/2,\mu+\kappa+1/2)} \int_{0}^{1} t^{\mu-\kappa-1/2}(1-t)^{\mu+\kappa-1/2} \exp(zt)\,\mathrm{d}t,$$

$$M_{\kappa,\mu}(z) = \frac{(\beta-\alpha)^{-2\mu} z^{\mu+1/2}\exp\left(-z/2\right)}{B(\mu-\kappa+1/2,\mu+\kappa+1/2)} \times \int_{\alpha}^{\beta} (u-\alpha)^{\mu-\kappa-1/2}(\beta-u)^{\mu+\kappa-1/2} \exp\left[\frac{z(u-\alpha)}{\beta-\alpha}\right] \mathrm{d}u,$$

$$M_{\kappa,\mu}(z) = \frac{2^{-2\mu} z^{\mu+1/2}}{B(\mu-\kappa+1/2,\mu+\kappa+1/2)} \times \int_{-1}^{1} (1+u)^{\mu-\kappa-1/2}(1-u)^{\mu+\kappa-1/2} \exp\left(\frac{zu}{2}\right) \mathrm{d}u.$$

and

$$M_{\kappa,\mu}(z) = \frac{z^{\mu+1/2}\exp(-z/2)}{B(\mu-\kappa+1/2,\mu+\kappa+1/2)} \int_{0}^{\infty} \frac{u^{\mu-\kappa-1/2}\exp[z(1+u)^{-1}u]}{(1+u)^{2\mu+1}} \mathrm{d}u. \quad (25)$$

Replacing $\exp(-\sigma/t)$ and $\exp[-\sigma/(1-t)]$ by their respective series expansions involving Laguerre polynomials $L_n(\sigma)\equiv L_n^{(0)}(\sigma)\,(n=0,1,2,\ldots)$ given in Miller [9, Eq. 3.4a, 3.4b], namely,

$$\exp\left(-\frac{\sigma}{t}\right) = \exp(-\sigma)t\sum_{n=0}^{\infty} L_n(\sigma)(1-t)^n, \quad |t|<1,$$

and

$$\exp\left(-\frac{\sigma}{1-t}\right) = \exp(-\sigma)(1-t)\sum_{m=0}^{\infty} L_m(\sigma)t^m, \quad |t|<1,$$

in (21), and integrating t using (3), the extended Whittaker function can also be expressed as

$$M_{\sigma,\kappa,\mu}(z)=\frac{2^{-2\mu}\exp(-2\sigma)z^{\mu+1/2}}{B(\mu-\kappa+1/2,\mu+\kappa+1/2)}\sum_{m,n=0}^{\infty}B\left(\mu-\kappa+\frac{3}{2}+m,\mu+\kappa+\frac{3}{2}+n\right)$$
$$\times L_m(\sigma)L_n(\sigma)\Phi\left(\mu-\kappa+\frac{3}{2}+m;2\mu+m+n+3;z\right).$$

If we put $z=0$ in (21) and compare the resulting expression with (7), we obtain an interesting relationship between the extended Whittaker function and extended beta function

$$M_{\sigma,\kappa,\mu}(0)=\frac{z^{\mu+1/2}\exp\left(-z/2\right)B(\mu-\kappa+1/2,\mu+\kappa+1/2;\sigma)}{B(\mu-\kappa+1/2,\mu+\kappa+1/2)}.$$

Theorem 3.1. *For $\sigma\geq 0$,*

$$M_{\sigma,\kappa,\mu}(-z)=(-1)^{\mu+1/2}M_{\sigma,-\kappa,\mu}(z).$$

Proof. Using the transformation (16) in (20) we have

$$M_{\sigma,\kappa,\mu}(-z)=(-1)^{\mu+1/2}z^{\mu+1/2}\exp\left(-\frac{z}{2}\right)\Phi_\sigma\left(\mu+\kappa+\frac{1}{2};2\mu+1;z\right).$$

Now, writing the right hand side of the above expression in terms of extended Whittaker function by using (20), we get the result. □

Theorem 3.2. *If $\sigma>0$, $\mu>-1/2$ and $\mu\pm\kappa>-1/2$, then*

$$M_{\sigma,\kappa,\mu}(z)\leq\exp(-4\sigma)M_{\kappa,\mu}(z)\leq\frac{\exp(-1)}{4\sigma}M_{\kappa,\mu}(z).$$

Proof. For $u>0$ and $\sigma>0$, $\sigma(u+u^{-1}-2)\geq 0$ implies that $\sigma(u+u^{-1})\geq 2\sigma$ and $\exp[-\sigma(u+u^{-1})]\leq\exp(-2\sigma)$.

Now, using this inequality in the representation given in (24), we get

$$M_{\sigma,\kappa,\mu}(z)\leq\frac{\exp(-4\sigma)z^{\mu+1/2}\exp\left(-z/2\right)}{B(\mu-\kappa+1/2,\mu+\kappa+1/2)}\int_0^\infty\frac{u^{\mu-\kappa-1/2}}{(1+u)^{2\mu+1}}\exp\left[\frac{zu}{1+u}\right]\mathrm{d}u$$
$$=\exp(-4\sigma)M_{\kappa,\mu}(z),$$

where the last line has been obtained by using (25). Further, the inequality $\ln v\leq v-1$, $v>0$, for $v=4\sigma$, yields

$$\exp(-4\sigma)\leq\frac{\exp(-1)}{4\sigma},$$

which gives the second part of the inequality. □

Using the inequality $\exp(x)>1+x^n/n!$, $x>0$, $n>0$, in (21), anthor interesting inequality is obtained as

$$M_{\sigma,\kappa,\mu}(z) \geq \frac{z^{\mu+1/2}\exp(-z/2)}{B(\mu-\kappa+1/2,\mu+\kappa+1/2)} \times \int_0^1 t^{\mu-\kappa-1/2}(1-t)^{\mu+\kappa-1/2}\left[1+\frac{(zt)^n}{n!}\right]\exp\left[-\frac{\sigma}{t(1-t)}\right]\mathrm{d}t.$$

Now, evaluating the above integral by using (7), we get

$$M_{\sigma,\kappa,\mu}(z) \geq \frac{z^{\mu+1/2}\exp(-z/2)}{B(\mu-\kappa+1/2,\mu+\kappa+1/2)}\left[B\left(\mu-\kappa+\frac{1}{2},\mu+\kappa+\frac{1}{2};\sigma\right) + \frac{z^n}{n!}B\left(\mu-\kappa+n+\frac{1}{2},\mu+\kappa+\frac{1}{2};\sigma\right)\right].$$

The following theorem gives the Mellin transform of the extended Whittaker function.

Theorem 3.3. *For* $\mathrm{Re}(s) > 0$ *and* $\mathrm{Re}(\mu \pm \kappa + s) > -1/2$*, we have*

$$\int_0^\infty \sigma^{s-1}M_{\sigma,\kappa,\mu}(z)\,\mathrm{d}\sigma = \frac{\Gamma(s)B(\mu-\kappa+s+1/2,\mu+\kappa+s+1/2)}{B(\mu-\kappa+1/2,\mu+\kappa+1/2)}z^{-s}M_{\kappa,\mu+s}(z).$$

Proof. Replacing $M_{\sigma,\kappa,\mu}(z)$ by its definition given in (20), one obtains

$$\int_0^\infty \sigma^{s-1}M_{\sigma,\kappa,\mu}(z)\,\mathrm{d}\sigma = z^{\mu+1/2}\exp\left(-\frac{z}{2}\right)\int_0^\infty \sigma^{s-1}\Phi_\sigma\left(\mu-\kappa+\frac{1}{2};2\mu+1;z\right)\mathrm{d}\sigma.$$

Now, evaluating the above integral by using (17) and then writing the resulting expression in terms of Whittaker function, we get the final result. □

Substitution of $s = 1$ in the previous theorem yields an interesting relationship between the functions $M_{\sigma,\mu,\kappa}(z)$ and $M_{\mu,\kappa}(z)$ as

$$\int_0^\infty M_{\sigma,\kappa,\mu}(z)\,\mathrm{d}\sigma = \frac{B(\mu-\kappa+3/2,\mu+\kappa+3/2)}{B(\mu-\kappa+1/2,\mu+\kappa+1/2)}z^{-1}M_{\kappa,\mu+1}(z).$$

Theorem 3.4. *If* $\sigma \geq 0$, $2\alpha > \beta > 0$, *and* $\mathrm{Re}(a+\mu) > -1/2$*, then*

$$\int_0^\infty \exp(-\alpha x)x^{a-1}M_{\sigma,\kappa,\mu}(\beta x)\,\mathrm{d}x = \frac{\Gamma(a+\mu+1/2)\beta^{\mu+1/2}}{(\alpha+\beta/2)^{a+\mu+1/2}}F_\sigma\left(a+\mu+\frac{1}{2},\mu-\kappa+\frac{1}{2};2\mu+1;\frac{2\beta}{2\alpha+\beta}\right). \tag{26}$$

Proof. Using the integral representation of the extended Whittaker function given in (21), we have

$$\int_0^\infty \exp(-\alpha x)x^{a-1}M_{\sigma,\kappa,\mu}(\beta x)\,\mathrm{d}x = \frac{\beta^{\mu+1/2}}{B(\mu-\kappa+1/2,\mu+\kappa+1/2)}\int_0^1 t^{\mu-\kappa-1/2}(1-t)^{\mu+\kappa-1/2}\exp\left[-\frac{\sigma}{t(1-t)}\right] \times \int_0^\infty x^{a+\mu-1/2}\exp\left[-\left(\alpha+\frac{1}{2}\beta-\beta t\right)x\right]\mathrm{d}x\,\mathrm{d}t.$$

Now, first integrating x by using the definition of the gamma function then integrating t by applying (10), we get the desired result. □

Corollary 3.4.1. *If* $\sigma \geq 0$*,* $\alpha > 0$*, and* $\mathrm{Re}(a+\mu) > -1/2$*, then*

$$\int_0^\infty \exp(-\alpha x) x^{a-1} M_{\sigma,\kappa,\mu}(2\alpha x)\,\mathrm{d}x$$
$$= \frac{\Gamma(a+\mu+1/2)B(\mu-\kappa+1/2,\kappa-a;\sigma)}{B(\mu-\kappa+1/2,\mu+\kappa+1/2)(2\alpha)^a}.$$

Proof. Take $\beta = 2\alpha$ in (26), then use (14) in the resulting expression. □

4. EXTENDED TRICOMI'S FUNCTION

In this section, we give an extended form of the Tricomi's function or confluent hypergeometric function of the second kind and show that this function occurs naturally in statistical distribution theory.

The extended confluent hypergeometric function of the second kind, denoted by $\Psi_\sigma(b;c;z)$, may be defined as

$$\Psi_\sigma(b;c;z) = \frac{1}{\Gamma(b)} \int_0^\infty t^{b-1}(1+t)^{c-b-1} \exp\left(-zt - \frac{\sigma}{t}\right)\,\mathrm{d}t, \quad \sigma \geq 0, \tag{27}$$

where $z > 0$ and $\mathrm{Re}(b) > 0$. For $\sigma = 0$, this function reduces to the classical confluent hypergeometric function of the second kind defined by the integral

$$\Psi(b;c;z) = \frac{1}{\Gamma(b)} \int_0^\infty t^{b-1}(1+t)^{c-b-1} \exp\left(-zt\right)\,\mathrm{d}t,$$

where $z > 0$ and $\mathrm{Re}(b) > 0$. By making the substitution $zt = v$ in (27), the extended confluent hypergeometric function of the second kind can also be represented as

$$\Psi_\sigma(b;c;z) = \frac{z^{-b}}{\Gamma(b)} \int_0^\infty v^{b-1} \left(1+\frac{v}{z}\right)^{c-b-1} \exp\left(-v - \frac{\sigma z}{v}\right)\,\mathrm{d}v. \tag{28}$$

For $c = b+1$, (28) slides to

$$\Psi_\sigma(b;b+1;z) = \frac{z^{-b}}{\Gamma(b)} \int_0^\infty v^{b-1} \exp\left(-v - \frac{\sigma z}{v}\right)\,\mathrm{d}v$$
$$= \frac{z^{-b}}{\Gamma(b)} \Gamma(b;\sigma z), \quad \sigma z \geq 0, \tag{29}$$

where $\Gamma(b;\sigma)$ is the extended gamma function. Also, substituting $c = b+1$ in (27) and comparing the resulting expression with (12), we obtain a direct relationship between the extended confluent hypergeometric function of the second kind and the modified Bessel function of the second kind as

$$\Psi_\sigma(b;b+1;z) = \frac{2}{\Gamma(b)} \left(\frac{\sigma}{z}\right)^{b/2} K_b(2\sqrt{\sigma z}), \quad \sigma > 0, \quad z > 0. \tag{30}$$

Note that, in view of (13), the expressions (29) and (30) are equivalent if z and σ are positive.

The Mellin transform of the extended confluent hypergeometric function of the second kind is derived as

$$\int_0^\infty \sigma^{s-1} \Phi_\sigma(b;c;z)\,\mathrm{d}\sigma = \frac{\Gamma(s)\Gamma(b+s)}{\Gamma(b)} \Psi(b+s;c+s;z),$$

where $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(b) > 0$.

Finally, we give the following theorem which gives the density of the ratio of two independent random variables in terms of extended confluent hypergeometric function of the second kind.

Theorem 4.1. *If $U \sim \mathrm{Ga}(a, \sigma_1)$ and $V \sim \mathrm{EB2}(b, c; \sigma_2)$ are independent, then the density of $X = U/V$ is given by*

$$\frac{\Gamma(a+b)x^{a-1}}{\sigma_1^a \Gamma(a) B(b, c; \sigma_2) \exp(2\sigma_2)} \Psi_{\sigma_2}\left(a+b, a-c+1; \sigma_2 + \frac{x}{\sigma_1}\right), \quad x > 0,$$

where $\sigma_1 > 0$, $\sigma_2 > 0$, $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(a+b) > 0$.

Proof. As U and V are independent, from (18) and (19), the joint density of U and V is given by

$$\frac{u^{a-1}v^{b-1}(1+v)^{-(b+c)} \exp[-u/\sigma_1 - \sigma_2(v + 1/v)]}{\sigma_1^a\, \Gamma(a) B(b, c; \sigma_2) \exp(2\sigma_2)}, \quad u > 0, \quad v > 0.$$

Making the transformation $X = U/V$, with the Jacobian $J(u \to x) = v$, we find the joint density of V and X as

$$\frac{x^{a-1}v^{a+b-1}(1+v)^{-(b+c)} \exp[-xv/\sigma_1 - \sigma_2(v + 1/v)]}{\sigma_1^a \Gamma(a) B(b, c; \sigma_2) \exp(2\sigma_2)}, \quad v > 0, \quad x > 0.$$

Now, the density of X is obtained by integrating the above expression with respect to v as

$$\frac{x^{a-1}}{\sigma_1^a \Gamma(a) B(b, c; \sigma_2) \exp(2\sigma_2)} \int_0^1 \frac{v^{a+b-1}}{(1+v)^{b+c}} \exp\left[-\left(\sigma_2 + \frac{x}{\sigma_1}\right) v - \frac{\sigma_2}{v}\right] \mathrm{d}v.$$

Finally, evaluating the above integral by using the integral representation (27), we obtain the desired result. □

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